Transport equations with singularity

H.Babovsky, G.N.Milstein
Weierstrass Institute for Applied Analysis and Stochastic
Berlin, Germany

Abstract

Test particle dynamics in the vicinity of singularities of a scattering medium are important for a number of physical applications, one of which is the coupling of kinetic and continuum equations in numerical simulations. Out of several relevant questions arising in this context, the paper concerns boundary conditions for the related hyperbolic system of equations in a slab between two singularities. For a linear model problem, we investigate first exit times from the slab and give a complete characterization of the stochastic particle dynamics, which provides a classification of the hyperbolic systems into Cauchy problems and into those which have to be supplemented with boundary conditions. Properties of the corresponding process like ergodicity, recurrence and asymptotic behaviour are investigated.

Key words: Kinetic equations, Markov process, probabilistic representation for PDE, ergodicity.

1. Introduction

Monte Carlo simulations by now have established as the main tool for the numerical solution of linear and nonlinear kinetic equations, and their mathematical structure is well understood (see[3]). An important question concerning numerics for kinetic equations via Monte Carlo simulations is that of the coupling to continuum flows. The importance arises from (at least) two reasons. First, the complexity of continuum equations is lower than that for kinetic equations – due to the lack of integrations over velocity space. Second, continuum regimes are often characterized by large collision frequencies of gas particles. Simulating these with Monte Carlo techniques for kinetic equations requires small time steps and with this large computational effort. A couple of recent publications by several scientists reflects the effort of coupling kinetic equations (in regimes where these are required) to continuum equations (used in regimes where these equations provide a sufficient description). An example for the coupling of nonlinear equations is given in [4]. In most cases, coupling techniques are developed by numerical experiments; theoretical investigations are by now developed not very far.

Continuum equations are derived by introducing a singularity into the kinetic equation. Consider the (for simplicity one-dimensional) kinetic equation for the density function f in phase space (with x as position and v as velocity vector)

$$(\partial_t + \lambda_1(v-u)\partial_x)f(t,x,v) = \lambda_2 J(f)$$

with appropriately chosen collision operator J (the precise form of which is here of no importance) and some vector u. Keeping λ_1 constant while rescaling λ_2 like $1/\varepsilon$ leads to a hyperbolic equation in the limit $\varepsilon \to 0$, while $\lambda_1 \sim 1/\varepsilon$ and $\lambda_2 \sim 1/\varepsilon^2$ results in a parabolic equation. (The latter case only holds if u is properly chosen as a certain moment of the unique eigenfunction of the collision operator J, see[1]). Results like these are described in a stochastic setting e.g. by Papanicolaou [17]. An example in a functional analytic framework (which in some cases is easier to handle and provides additional results) was treated in [2]. For a short review, including some physical and numerical background, see [1]. For the coupling of these "continuum equations" to related kinetic equations, two cases are of particular interest.

Suppose that in the whole computational domain, the collision frequencies differ at one order (or more) of magnitude (which may happen in a number of applications like gas expansion into vacuum, gas centrifuges, gas condensation, etc.) and that in regions of high collision frequencies a continuum description is appropriate. This forces on the kinetic side to introduce a space dependent function λ_2 with singularities reflecting the high collision frequencies.

Consider the stationary version of the above kinetic equation and suppose that the situation requires a very fine grid refinement in a small region (like a boundary layer) for an appropriate numerics. This is described best by introducing a space transformation $x \to \xi(x)$ where ξ has a large gradient close to the critical regime. This transformation again leads to space dependent λ_i differing by orders of magnitude from one point to another.

When considering Monte Carlo particle systems for the numerical simulation of such situations, a deep understanding of particle dynamics close to the singularity is required. In particular, the probability for a particle to cross the singularity and the crossing times are important characteristics, as well as the existence of stationary kinetic solutions in domains bounded by singularities. The answers to these questions have immediate consequences e.g. for the formulation of coupling conditions between different domains. To our knowledge,there do not exist any investigations into these problems which are of great relevance for stochastic algorithms for transport problems. The present paper is a first step into the investigation of stochastic particle dynamics in the vicinity of singularities of the collision frequency. For simplicity, we restrict to one space dimension and kinetic dynamics described by a two-velocity model.

The scope of the paper is as follows. In Section 2 we introduce a linear kinetic 2-velocity model equation for a test particle in a slab $[l_o, r_o]$ with singularities at l_o and r_o. The corresponding stochastic dynamics is that of a particle driven by a velocity which is a Markov process. The primary question is whether (dependent on the type of the singularities) the related system of hyperbolic equations is to be treated as a Cauchy problem or as a boundary value problem. The answer is intimately connected to the question of finiteness or infiniteness of first exit times from the slab. In Section 3, sufficient and necessary conditions for these are derived. In Sections 4 and 5, properties of the corresponding process like ergodicity, recurrence and asymptotic behavior are investigated and a complete classification of the hyperbolic systems as Cauchy problems or boundary value problems is given. Two examples in Section 6 conclude the paper.

2. Probabilistic representation of solutions for systems of hyperbolic equations

Consider a system of ordinary differential equations interacting with a Markov chain

$$\frac{dX}{ds} = a(X, Z) \tag{2.1}$$

where $X = (X_1, ..., X_n)^\top$, $a(x, z) = (a_1(x, z), ..., a_n(x, z))^\top$ are $n-$dimensional vectors, Z is the Markov chain with m states $z_1, ..., z_m$. Let the functions $a_{ij}(x) = a_i(x, z_j)$, $i = 1, ..., n$, $j = 1, ...m$, $x \in R^n$, satisfy Lipshitz condition and grow at infinity not faster than a linear function of x. We assume the infinitesimal matrix Q of the chain Z to depend on the state x of X :

$$Q = \begin{bmatrix} -q_1(x) & q_{12}(x) & ... & q_{1m}(x) \\ q_{21}(x) & -q_2(x) & ... & q_{2m}(x) \\ . & . & . & . \\ q_{m1}(x) & q_{m2}(x) & ... & -q_m(x) \end{bmatrix}$$

where $q_i(x)$, $q_{ij}(x)$ are continuous nonnegative bounded in R^n functions and the relation

$$\sum_{j\neq i} q_{ij}(x) = q_i(x) \tag{2.2}$$

is fulfilled. Then the system (2.1) generates a Markov process (X, Z) with infinitesimal generator

$$Af(x, z_i) = \sum_{k=1}^{n} a_k(x, z_i)\frac{\partial f}{\partial x_k}(x, z_i) - q_i(x)f(x, z_i) + \sum_{j\neq i} q_{ij}(x)f(x, z_j),\ i = 1, ..., m,\ x \in R^n \tag{2.3}$$

The use of (2.3) makes possible to obtain a probabilistic representation for the solution of a Cauchy problem for systems of hyperbolic equations

$$\frac{\partial u_i}{\partial t} = \sum_{k=1}^{n} a_k(x, z_i)\frac{\partial u_i}{\partial x_k} + b(x, z_i)u_i - q_i(x)u_i$$

$$+\sum_{j\neq i} q_{ij}(x)u_j + c(x, z_i),\ i = 1, ..., m,\ x \in R^n,\ t > 0 \tag{2.4}$$

$$u_i(0, x) = f(x, z_i),\ i = 1, ..., m \tag{2.5}$$

The representation has a form

$$u_i(t, x) = Ef(X_{x,z_i}(t), Z_{x,z_i}(t)) \exp(\int_0^t b(X_{x,z_i}(s), Z_{x,z_i}(s))ds)$$

$$+E\int_0^t \exp(\int_0^s b(X_{x,z_i}(\vartheta), Z_{x,z_i}(\vartheta))d\vartheta)c(X_{x,z_i}(s), Z_{x,z_i}(s))ds,\ i = 1, ..., m,\ x \in R^n,\ t \geq 0 \tag{2.6}$$

where $X_{x,z_i}(s), Z_{x,z_i}(s)$ is a realization of the Markov process (X, Z) starting from $X(0) = x$, $Z(0) = z_i$.

The formula (2.6) is valid if, for instance, the functions $f_i(x) = f(x, z_i)$, $b_i(x) = b(x, z_i)$, $c_i(x) = c(x, z_i)$, and the partial derivatives $\dfrac{\partial f_i}{\partial x_j}(x)$, $i = 1, ..., m$, $j = 1, ..., n$, are continuous bounded functions in R^n.

Let an initial distribution of (X, Z) be given by density $\lambda_i(x)$, $i = 1, ..., m$, i.e.

$$P\{X(0) \in H,\ Z(0) = z_i\} = \int_H \lambda_i(x)dx$$

Let $\lambda_i(x)a_k(x, z_i)$ have continuous partial derivatives $\dfrac{\partial(\lambda_i a_k)}{\partial x_k}$ and integrals

$$I_i = \int_{R^n} \sum_{k=1}^{n} \frac{\partial(\lambda_i(x)a_k(x, z_i))}{\partial x_k} dx,\ i = 1, ..., m \tag{2.7}$$

absolutely converge. Then the density $\psi_i(s, x)$ of (X, Z) at time s satisfies the following system of differential equations

$$\frac{\partial \psi_i}{\partial s} = -\sum_{k=1}^{n} \frac{\partial(\psi_i a_k(x, z_i))}{\partial x_k} - q_i(x)\psi_i + \sum_{j \neq i} q_{ji}(x)\psi_j \tag{2.8}$$

with initial conditions

$$\psi_i(0, x) = \lambda_i(x),\ i = 1, ..., m \tag{2.9}$$

As a consequence we obtain that $\lambda_i(x)$ describes a stationary distribution, provided the integrals (2.7) absolutely converge, if and only if it satisfies the following system

$$-\sum_{k=1}^{n} \frac{\partial(\lambda_i a_k(x, z_i))}{\partial x_k} - q_i(x)\lambda_i + \sum_{j \neq i} q_{ji}(x)\lambda_j = 0,\ i = 1, ..., m \tag{2.10}$$

It should be noted that all what has been said above is correct if we take some open set $\mathcal{D} \subseteq R^n$ provided $X_{x,z_i}(t) \in \mathcal{D}$, for any $x \in \mathcal{D}$, $i = 1, ..., m$, $t > 0$.

Let us make some observations of bibliographical nature. The first probabilistic representation of the solution for hyperbolic equation (for telegraph equation) belongs to M.Kac. The sufficiently general systems of ordinary differential equations acting by a Markov chain were considered in [12]. The detailed description of the process (X, Z) was done in [14]. In a lot of papers such a process was treated in connection with random evolutions (see, for instance, [7], [9], [8] and references there). In all of these papers the process Z does not depend on a state of the process X. The interaction of general processes X and Z is considered in [15]. Instead of the system of ordinary differential equations (2.1) it is possible to examine a system of stochastic differential equations interacting with a Markov chain as well. Both Cauchy problems and boundary value problems for systems of partial differential equations arising in connection with interacting Markov processes are considered in [16].

Let us now pass to a boundary value problem. Here we restrict to a system of two hyperbolic equations concerning functions $u_1(t, x)$ and $u_2(t, x)$ where x is a one-dimensional variable. Thus the system (2.1) is an one-dimensional equation and the chain Z has two states. Let us denote $a(x, z_1) = a_1(x)$, $a(x, z_2) = a_2(x)$. Due to (2.2) we have

$$q_{12}(x) = q_1(x),\ q_{21}(x) = q_2(x)$$

Consider the system

$$\frac{\partial u_1}{\partial t} + a_1(x)\frac{\partial u_1}{\partial x} + (b_1(x) - q_1(x))u_1 + q_1(x)u_2 + c_1(x) = 0$$

$$\frac{\partial u_2}{\partial t}+a_2(x)\frac{\partial u_2}{\partial x}+q_2(x)u_1+(b_2(x)-q_2(x))u_2+c_2(x)=0,\ t_0\le t<t_1,\ l_o<x<r_o \tag{2.11}$$

Introduce initial

$$u_i(t_1,x)=f_i(t_1,x),\ i=1,2,\ l_o\le x\le r_o \tag{2.12}$$

and boundary conditions

$$u_1(t,r_o)=f_1(t,r_o),\ u_2(t,l_o)=f_2(t,l_o),\ t_0\le t<t_1 \tag{2.13}$$

It is assumed that the functions $f_i(t,x)$ are continuous in their domain of definition and

$$q_1(x)>0,\ q_2(x)>0,\ a_1(x)>0,\ a_2(x)<0,\ l_o\le x\le r_o \tag{2.14}$$

Denote $b(x,z_i)=b_i(x)$, $c(x,z_i)=c_i(x)$, $f(t,x,z_i)=f_i(t,x)$. Define the first exit time

$$\tau_{t,x,z_i}[l,r]=\inf\{s: X_{t,x,z_i}(s)\notin[l,r]\}$$

where $l\le x\le r$ and $X_{t,x,z_i}(s)$, $Z_{t,x,z_i}(s)$, $s\ge t$, is a realization of the Markov process (X,Z) starting from $X(t)=x$, $Z(t)=z_i$.

Then the solution of the problem (2.11)-(2.14) has the following probabilistic representation

$$u_i(t,x)=Ef(\tau\wedge t_1,X_{t,x,z_i}(\tau\wedge t_1),Z_{t,x,z_i}(\tau\wedge t_1))\exp(\int_t^{\tau\wedge t_1}b(X_{t,x,z_i}(s),Z_{t,x,z_i}(s))ds)$$

$$+E\int_t^{\tau\wedge t_1}c(X_{t,x,z_i}(s),Z_{t,x,z_i}(s))\exp(\int_t^s b(X_{t,x,z_i}(\vartheta),Z_{t,x,z_i}(\vartheta))d\vartheta)ds \tag{2.15}$$

where $\tau=\tau_{t,x,z_i}[l_o,r_o]$.

Of course, due to homogeneity of (X,Z) and due to independence of b and c on time we can start at the zero instant, put $t=0$ in (2.15) and take $X_{x,z_i}(s)$, $Z_{x,z_i}(s)$, $\tau=\tau_{x,z_i}[l_o,r_o]$, $\tau\wedge(t_1-t)$ instead of $X_{t,x,z_i}(s)$, $Z_{t,x,z_i}(s)$, $\tau=\tau_{t,x,z_i}[l_o,r_o]$, $\tau\wedge t_1$ in (2.15).

The mentioned probabilistic representation can be strongly obtained with help of Dynkin's formula [5]

$$Ef(Y_y(\tau))-f(y)=E\int_0^\tau Af(Y_y(s))ds,\ g=Af \tag{2.16}$$

which is valid under broad assumptions concerning some homogeneous Feller Markov process Y with infinitesimal generator A, $\tau=\tau_y$ is a Markov moment for Y, $E\tau_y<\infty$.

We pay attention that owing to the employment of the probabilistic representation the initial conditions (2.12) in the boundary value problem (2.11)-(2.14) are given on the right end point of the considered time interval in contrast to tradition to give initial condition on the left end point. But any traditional problem can be reduced by conversion of time to the problem in which initial conditions pass to the right end.

The representation (2.15) substantially makes use of conditions (2.14). If the functions $q_1(x)$, $q_2(x)$, $a_1(x)$, $-a_2(x)$ remain continuous and strongly positive only on the interval $l_o<x<r_o$ and if at l_o or r_o some of them receive the values 0 or ∞ then the singular cases can arise (for example, the situation is possible that the process X cannot attain the ends of interval (l_o,r_o) for finite time and the boundary conditions (2.13) become senseless). In such cases the statement of the problem depends on the

behavior of the process X on the interval (l_o, r_o). The definition of the process on the interval (l_o, r_o) with time life

$$\tau_{x,z_i}(l_o, r_o) = \sup\{\tau_{x,z_i}[l,r] : \; l_o < l < r < r_o\}$$

can be done in the usual way. Our approach to the behavior in many respects repeats the well known investigation of the one-dimensional diffusion on the bounded interval due to Feller (see, for instance, [5], [11]). We are interested in questions when it is possible to examine a boundary value problem for the system (2.11) and when a Cauchy problem. Concerning methodology we follow the exposition of one-dimensional diffusion in [6] and [10]. In the case when the ends l_o and r_o are unattainable in finite time for the process X we are interested in the existence of a stationary distribution and in ergodic characteristics of the process with their consequences for the system of hyperbolic equations. Here we follow [13], where such questions have been considered for diffusion equations.

3. The attainment probability and mean value of the first exit time

So we treat the scalar equation

$$\frac{dX}{dt} = a(X, Z)$$

where $X \in (l_o, r_o)$, the Markov chain Z takes two values z_1 and z_2, and $a_1(x) = a(x, z_1) > 0$, $a_2(x) = a(x, z_2) < 0$, $q_1(x) > 0$, $q_2(x) > 0$ on the whole interval (l_o, r_o). We assume the functions $a_1(x)$, $a_2(x)$ to satisfy Lipshitz conditions and the functions $q_1(x)$, $q_2(x)$ to be continuous on (l_o, r_o). Let $[l, r] \in (l_o, r_o)$ and $l \leq x \leq r$. On the closed interval $[l, r]$ all the functions a_1, a_2, q_1, q_2 are bounded and bounded away from zero. In that case it is possible to prove that $\tau_{x,z_i}[l, r] < \infty$ with probability 1 and $E\tau_{x,z_i}[l, r] < \infty$.

We shall find the probabilities

$$p_i(x) = p_i(x; l, r) = P(X_{x,z_i}(\tau_{x,z_i}[l, r]) = l), \; i = 1, 2$$

$$\bar{p}_i(x) = \bar{p}_i(x; l, r) = P(X_{x,z_i}(\tau_{x,z_i}[l, r]) = r) = 1 - p_i(x; l, r), \; i = 1, 2$$

The first of them is the probability that $X_{x,z_i}(t)$ attains l earlier than r and the second one that $X_{x,z_i}(t)$ attains r earlier than l. Note if $X_{x,z_i}(\tau_{x,z_i}[l, r]) = l$ (correspondingly $X_{x,z_i}(\tau_{x,z_i}[l, r]) = r$) then $Z_{x,z_i}(\tau_{x,z_i}[l, r]) = z_2$ (correspondingly $Z_{x,z_i}(\tau_{x,z_i}[l, r]) = z_1$) and $\tau_{l,z_2}[l, r] = 0$, $\tau_{r,z_1}[l, r] = 0$. Therefore $p_1(r) = 0$, $p_2(l) = 1$ and $\bar{p}_1(r) = 1$, $\bar{p}_2(l) = 0$. Let us use Dynkin's formula (2.16) taking in it as Y the process (X, Z). If we put $g = 0$, i.e. $Af = 0$, and $f(r, z_1) = 0$, $f(l, z_2) = 1$ then

$$Ef(X_{x,z_i}(\tau_{x,z_i}[l, r]), Z_{x,z_i}(\tau_{x,z_i}[l, r])) = P(X_{x,z_i}(\tau_{x,z_i}[l, r]) = l) = p_i(x)$$

and the formula (2.15) gives us

$$f(r, z_i) = P(X_{x,z_i}(\tau_{x,z_i}[l, r]) = l) = p_i(x)$$

Hence the desired probability $p_i(x)$ is the solution of the boundary value problem (we take $f_1(x) = f(x, z_1),\ f_2(x) = f(x, z_2)$)

$$a_1(x)\frac{\partial f_1}{\partial x} - q_1(x)f_1 + q_1(x)f_2 = 0$$

$$a_2(x)\frac{\partial f_2}{\partial x} - q_2(x)f_2 + q_2(x)f_1 = 0 \tag{3.1}$$

with boundary condition

$$f_1(r) = 0,\ \ f_2(l) = 1 \tag{3.2}$$

Analogously the probability $\bar{p}_i(x)$ satisfies the same system (3.1) but with boundary conditions

$$f_1(r) = 1,\ \ f_2(l) = 0 \tag{3.3}$$

Of course $\bar{p}_i(x)$ can be found from equality $\bar{p}_i(x) = 1 - p_i(x)$. Denote

$$k_1(x) = \frac{q_1(x)}{a_1(x)},\ \ k_2(x) = \frac{q_2(x)}{a_2(x)},\ \ k(x) = k_1(x) + k_2(x)$$

Note that $k_1(x) > 0$ and $k_2(x) < 0$ on the whole interval (l_o,r_o). Solving the problems (3.1)-(3.2) and (3.1)-(3.3) we obtain the following theorem.

Theorem 3.1. *Let $l_o < l < r < r_o$ and $l \le x \le r$. Then the following formulae are valid*

$$p_1(x) = p_1(x; l, r) = \frac{\int_x^r k_1(\xi)\exp(-\int_\xi^r k(\varsigma)d\varsigma)d\xi}{1 - \int_l^r k_2(\xi)\exp(-\int_\xi^r k(\varsigma)d\varsigma)d\xi} \tag{3.4}$$

$$p_2(x) = p_2(x; l, r) = \frac{1 - \int_x^r k_2(\xi)\exp(-\int_\xi^r k(\varsigma)d\varsigma)d\xi}{1 - \int_l^r k_2(\xi)\exp(-\int_\xi^r k(\varsigma)d\varsigma)d\xi} \tag{3.5}$$

$$\bar{p}_1(x) = \bar{p}_1(x; l, r) = 1 - p_1(x) = \frac{1 + \int_l^x k_1(\xi)\exp(\int_l^\xi k(\varsigma)d\varsigma)d\xi}{1 + \int_l^r k_1(\xi)\exp(\int_l^\xi k(\varsigma)d\varsigma)d\xi} \tag{3.6}$$

$$\bar{p}_2(x) = \bar{p}_2(x; l, r) = 1 - p_2(x) = \frac{-\int_l^x k_2(\xi)\exp(\int_l^\xi k(\varsigma)d\varsigma)d\xi}{1 + \int_l^r k_1(\xi)\exp(\int_l^\xi k(\varsigma)d\varsigma)d\xi} \tag{3.7}$$

Let us pass now to the calculation of the mathematical expectations $E\tau_{x,z_i}[l, r]$. In the next theorem we give two kinds of formulae for the expectations. The formulae (3.8)-(3.10) are adjusted to an analysis of the behavior of the process X on the right end r_o and (3.11)-(3.13) are adjusted for l_o. Denote

$$m(x) = \frac{1}{a_1(x)} - \frac{1}{a_2(x)}$$

We have $m(x) > 0$ on the whole interval (l_o, r_o).

Theorem 3.2. *Let $l_o < l \le x \le r < r_o$. Then*

$$E\tau_{x,z_1}[l, r] = C[l, r]\int_x^r k_1(\xi)\exp(-\int_\xi^r k(\varsigma)d\varsigma)d\xi$$

$$+\int_x^r (\frac{1}{a_1(\xi)} - k_1(\xi)\int_\xi^r m(\eta)\exp(-\int_\xi^\eta k(\varsigma)d\varsigma)d\eta)d\xi \tag{3.8}$$

$$E\tau_{x,z_2}[l, r] = C[l, r](1 - \int_x^r k_2(\xi)\exp(-\int_\xi^r k(\varsigma)d\varsigma)d\xi)$$

$$+\int_x^r (\frac{1}{a_2(\xi)} + k_2(\xi)\int_\xi^r m(\eta)\exp(-\int_\xi^\eta k(\varsigma)d\varsigma)d\eta)d\xi \tag{3.9}$$

where

$$C[l,r] = E\tau_{r,z_2}[l,r] = \frac{\int_l^r (-\frac{1}{a_2(\xi)} - k_2(\xi)\int_\xi^r m(\eta)\exp(-\int_\xi^\eta k(\varsigma)d\varsigma)d\eta)d\xi}{1-\int_l^r k_2(\xi)\exp(-\int_\xi^r k(\varsigma)d\varsigma)d\xi} \tag{3.10}$$

and

$$E\tau_{x,z_1}[l,r] = \bar{C}[l,r](1+\int_l^x k_1(\xi)\exp(\int_l^\xi k(\varsigma)d\varsigma)d\xi)$$

$$-\int_l^x (\frac{1}{a_1(\xi)} + k_1(\xi)\int_l^\xi m(\eta)\exp(\int_\eta^\xi k(\varsigma)d\varsigma)d\eta)d\xi \tag{3.11}$$

$$E\tau_{x,z_2}[l,r] = -\bar{C}[l,r]\int_l^x k_2(\xi)\exp(\int_l^\xi k(\varsigma)d\varsigma)d\xi$$

$$-\int_l^x (\frac{1}{a_2(\xi)} - k_2(\xi)\int_l^\xi m(\eta)\exp(\int_\eta^\xi k(\varsigma)d\varsigma)d\eta)d\xi \tag{3.12}$$

where

$$\bar{C}[l,r] = E\tau_{l,z_1}[l,r] = \frac{\int_l^r (\frac{1}{a_1(\xi)} + k_1(\xi)\int_l^\xi m(\eta)\exp(\int_\eta^\xi k(\varsigma)d\varsigma)d\eta)d\xi}{1+\int_l^r k_1(\xi)\exp(\int_l^\xi k(\varsigma)d\varsigma)d\xi} \tag{3.13}$$

Proof. Let us use again the Dynkin's formula (2.16) taking in it as Y the process (X, Z). If we put $g = -1$, i.e. $Af = -1$, and $f(r, z_1) = 0$, $f(l, z_2) = 0$ then

$$Ef(X_{x,z_i}(\tau_{x,z_i}[l,r]), Z_{x,z_i}(\tau_{x,z_i}[l,r])) = 0$$

and the formula (2.15) gives

$$f_i(x) = f(x, z_i) = E\tau_{x,z_i}[l,r]$$

Thus the required mathematical expectation is the solution of the following boundary value problem

$$a_1(x)\frac{\partial f_1}{\partial x} - q_1(x)f_1 + q_1(x)f_2 = -1$$

$$a_2(x)\frac{\partial f_2}{\partial x} - q_2(x)f_2 + q_2(x)f_1 = -1$$

$$f_1(r) = 0,\ f_2(l) = 0$$

Formulae (3.8)-(3.13) are obtained by direct solution of this problem. Theorem 3.2 is proved.

4. A Cauchy problem or a boundary value problem depending on attainability of interval ends

Introduce integrals

$$I(l_o, r] = -\int_{l_o}^r k_2(\xi)\exp(-\int_\xi^r k(\varsigma)d\varsigma)d\xi,\ l_o < r < r_o \tag{4.1}$$

$$\bar{I}[l, r_o) = \int_l^{r_o} k_1(\xi) \exp(\int_l^{\xi} k(\varsigma)d\varsigma)d\xi, \ l_o < l < r_o \tag{4.2}$$

Note that both the boundedness of the integral (4.1) (or (4.2)) and the unboundedness of it does not depend on the choice of r (or l). The probability $p_i(x; l, r)$, $i = 1, 2$, monotonically decreases under $l \downarrow l_o$ and therefore it has a limit which is denoted as $p_i(x; l_o, r)$. Analogously, $\bar{p}_i(x; l_o, r)$, $p_i(x; l, r_o)$, $\bar{p}_i(x; l, r_o)$, denote the corresponding limits. It is clear that, for example, $p_i(x; l_o, r) = 0$ if $I(l_o, r] = \infty$ (see formulae (3.4)-(3.7)).

Theorem 4.1. *Let*

$$I(l_o, r] = \infty \tag{4.3}$$

Then for $l_o < x \le r$, $i = 1, 2$, the time $\tau_{x,z_i}(l_o, r]$ is finite with probability 1 *and the relation*

$$X_{x,z_i}(\tau_{x,z_i}(l_o, r]) = r$$

is also fulfilled with probability 1.

If

$$I(l_o, r] < \infty \tag{4.4}$$

then the event

$$\mathcal{B} = \left\{ \omega : \lim_{t \uparrow \tau_{x,z_i}(l_o, r]} X_{x,z_i}(t) = r \right\}$$

has the probability

$$0 < P(\mathcal{B}) = \bar{p}_i(x; l_o, r) < 1$$

and the event

$$\mathcal{A} = \left\{ \omega : \lim_{t \uparrow \tau_{x,z_i}(l_o, r]} X_{x,z_i}(t) = l_o \right\}$$

has the probability

$$P(\mathcal{A}) = p_i(x; l_o, r) = 1 - \bar{p}_i(x; l_o, r)$$

Besides for $\omega \in \mathcal{B}$ the time $\tau_{x,z_i}(l_o, r]$ is finite.

The analogous statement is just for $[l, r_o)$.

Proof. Let us consider the event

$$\mathcal{B}_l = \{\omega : \ X_{x,z_i}(\tau_{x,z_i}[l, r]) = r\}$$

under fixed x, z_i, r and $l_o < l < r$. Clearly, the sequence of the events grows with $l \downarrow l_o$, $\mathcal{B} = \cup_{l_o < l < r} \mathcal{B}_l$ and $P(\mathcal{B}) = \lim_{l \downarrow l_o} P(\mathcal{B}_l)$. Due to Theorem 3.1 $p_i(x; l_o, r) = 0$ in the case (4.3) and $0 < p_i(x; l_o, r) < 1$ in the case (4.4). Consequently $P(\mathcal{B}) = \bar{p}_i(x; l_o, r) = 1$ in the case (4.3) and it is equal to $0 < \bar{p}_i(x; l_o, r) < 1$ in the case (4.4). Further for $\omega \in \mathcal{B}_l$ the time $\tau_{x,z_i}(l_o, r]$ coincides with $\tau_{x,z_i}[l, r]$ and therefore for every $\omega \in \mathcal{B}$ it is finite. Thus it remains to prove only that $P(\mathcal{A}) = 1 - \bar{p}_i(x; l_o, r)$. Note that under $t \uparrow \tau_{x,z_i}(l_o, r]$ the $X_{x,z_i}(t)$ either has a limit, and then this limit is equal either l_o or r, or has not any limit. Let us prove that the event

$$\mathcal{C} = \{\omega : \ X_{x,z_i}(t) \text{ has no limit under } t \uparrow \tau_{x,z_i}(l_o, r]\}$$

has the probability 0. For every $\omega \in \mathcal{C}$ there exist two rational numbers $r_1 < r_2$ which belong to the interval (l_o, r) and such that

$$\liminf_{t \uparrow \tau_{x,z_i}(l_o, r]} X_{x,z_i}(t) \le r_1, \ \limsup_{t \uparrow \tau_{x,z_i}(l_o, r]} X_{x,z_i}(t) \ge r_2 \tag{4.5}$$

Prove that the probability of the event (4.5) is equal to 0. Indeed there exists the infinite sequence of Markov moments τ_n in which X attains r_2 after r_1. And the probability to attain r earlier than r_1 from r_2 is positive. Therefore the probability not to attain r for an infinite number of such steps is equal to 0, i.e., it is proved that the probability of the event (4.5) is equal to 0. But the event $\mathcal{C}$ belongs to the countable union of such events. So the equality $P(\mathcal{C}) = 0$ is proved. Hence $P(\mathcal{A} \cup \mathcal{B}) = 1$. Theorem 4.1 is proved.

Theorem 4.2. *If*

$$I(l_o, r] = \infty, \ \bar{I}[l, r_o) = \infty \tag{4.6}$$

then for every $l_o < x < r_o$, z_i, $i = 1, 2$,

$$P\left\{\lim \inf_{t \uparrow \tau_{x,z_i}(l_o,r_o)} X_{x,z_i}(t) = l_o\right\} = P\left\{\lim \sup_{t \uparrow \tau_{x,z_i}(l_o,r_o)} X_{x,z_i}(t) = r_o\right\} = 1 \tag{4.7}$$

the time $\tau_{x,z_i}(l_o, r_o)$ is equal to ∞ with probability 1, and the process (X, Z) is recurrent, i.e., (in our case) for any two points (x, z_i), (y, z_j), $x, y \in (l_o, r_o)$, $i, j = 1, 2$, the probability $P\left\{\tau_{x,z_i}^{y,z_j} < \infty\right\}$ where τ_{x,z_i}^{y,z_j} is the first time in which the process $(X_{x,z_i}(t), Z_{x,z_i}(t))$ attains (y, z_j) is equal to 1.

If

$$I(l_o, r] < \infty, \ \bar{I}[l, r_o) = \infty \ (I(l_o, r] = \infty, \ \bar{I}[l, r_o) < \infty) \tag{4.8}$$

then

$$P\left\{\lim_{t \uparrow \tau_{x,z_i}(l_o,r_o)} X_{x,z_i}(t) = l_o\right\} = 1 \ (P\left\{\lim_{t \uparrow \tau_{x,z_i}(l_o,r_o)} X_{x,z_i}(t) = r_o\right\} = 1) \tag{4.9}$$

If

$$I(l_o, r] < \infty, \ \bar{I}[l, r_o) < \infty \tag{4.10}$$

then

$$P\left\{\lim_{t \uparrow \tau_{x,z_i}(l_o,r_o)} X_{x,z_i}(t) = l_o\right\} = p_i(x; l_o, r_o) = \lim_{r \uparrow r_o} p_i(x; l_o, r) \tag{4.11}$$

$$P\left\{\lim_{t \uparrow \tau_{x,z_i}(l_o,r_o)} X_{x,z_i}(t) = r_o\right\} = \bar{p}_i(x; l_o, r_o) = \lim_{l \downarrow l_o} \bar{p}_i(x; l, r_o) = 1 - p_i(x; l_o, r_o) \tag{4.12}$$

Proof. Let $l_n \downarrow l_o$, $r_n \uparrow r_o$ and $l_1 < x < r_1$. Consider $(l_o, r_1]$. Due to Theorem 4.1 $X_{x,z_i}(t)$ attains r_1 with probability 1 in a finite time τ_1. Note that $X_{x,z_i}(\tau_1) = r_1$ and $Z_{x,z_i}(\tau_1) = z_1$. Now consider $[l_1, r_0)$. Again due to Theorem 4.1 $X_{r_1,z_1}(t)$ attains l_1 with probability 1 in a finite time τ_2. At the moment $\tau_1 + \tau_2$ the process (X, Z) is at the state l_1, z_2. In a finite time τ_3 our process will be at the state r_2, z_1 and so on. The relation (4.7) is proved. The infinity of $\tau_{x,z_i}(l_o, r_o)$ follows from the uniform boundedness below of all τ_n thanks to bounded velocity on the $[l_1, r_1]$. The recurrence property easily follows from foregoing reasoning. Let $I(l_o, r] < \infty$ and $I[l, r_o) = \infty$. Introduce the set

$$\mathcal{A}_r = \left\{\omega : \lim_{t \uparrow \tau_{x,z_i}(l_o,r]} X_{x,z_i}(t) = l_o\right\}$$

The probability of this set is equal to $P(\mathcal{A}_r) = p_i(x; l_o, r) = 1 - \bar{p}_i(x; l_o, r)$, and $\mathcal{A}_r$ increases under $r \uparrow r_o$. But $\bar{p}_i(x; l_o, r) < \bar{p}_i(x; l, r)$ for any $l_o < l < x$ and $\bar{p}_i(x; l, r) \downarrow 0$ under $r \uparrow r_o$ by virtue of (3.6)-(3.7) and the condition $I[l, r_o) = \infty$. Thus the second assertion of the theorem is proved. The formula (4.11) follows from equality

$\left\{\omega : \lim_{t\uparrow\tau_{x,z_i}(l_o,r_o)} X_{x,z_i}(t) = l_o\right\} = \cup\mathcal{A}_r$. The analogous equality gives the first part of the formula (4.12). By the same way as in Theorem 4.1 it is possible to prove that $\lim_{t\uparrow\tau_{x,z_i}(l_o,r_o)} X_{x,z_i}(t)$ exists with probability 1 and it is equal to either l_o or r_o. Hence $p_i(x; l_o, r_o) + \bar{p}_i(x; l_o, r_o) = 1$. Theorem 4.2 is proved.

Corollary. *The process (X, Z) is recurrent if and only if the condition (4.6) is fulfilled.*

Both in Theorem 4.1 and in Theorem 4.2 there was no indication whether the time τ is finite or infinite in the cases (4.4), (4.8) and (4.10). But this question is very important. For example, if for all (x, z_i) the time $\tau_{x,z_i}(l_o, r_o)$ is equal to ∞ with probability 1 then one can examine only a Cauchy problem for the system of hyperbolic equations (2.11). In reality the time τ can be either finite or infinite. Moreover it is either finite or infinite at once for all ω from the corresponding event.

First of all let us note that if $E\tau_{x,z_i}(l_o, r] < \infty$ then $\tau_{x,z_i}(l_o, r] < 1$ with probability 1. Therefore the next theorem is devoted to the case $E\tau_{x,z_i}(l_o, r] = \infty$. As follows easily from Theorem 3.2 finiteness or infinity of the $E\tau_{x,z_i}(l_o, r]$ does not depend on x, z_i, r, $l_o < r < r_o$, $l_o < x < r$, $i = 1, 2$ (of course, $E\tau_{x,z_i}(l_o, r] = \lim_{l\downarrow l_o} E\tau_{x,z_i}[l, r]$ since $\tau_{x,z_i}[l, r] \uparrow \tau_{x,z_i}(l_o, r]$ with $l \downarrow l_o$).

Introduce the integrals

$$J(l_o, r] = \int_{l_o}^{r} (-\frac{1}{a_2(\xi)} - k_2(\xi) \int_{\xi}^{r} m(\eta) \exp(-\int_{\xi}^{\eta} k(\varsigma)d\varsigma)d\eta)d\xi \tag{4.13}$$

$$\bar{J}[l, r_o) = \int_{l}^{r_o} (\frac{1}{a_1(\xi)} + k_1(\xi) \int_{l}^{\xi} m(\eta) \exp(\int_{\eta}^{\xi} k(\varsigma)d\varsigma)d\eta)d\xi \tag{4.14}$$

It is clear (see the formulae (3.8)-(3.10)) that under the condition $I(l_o, r] < \infty$ the integral $J(l_o, r]$ and the mathematical expectation $E\tau_{x,z_i}(l_o, r]$ are simultaneously either finite or infinite. The same is just with respect to $\bar{J}[l, r_o)$ and $E\tau_{x,z_i}[l, r_o)$ provided that $\bar{I}[l, r_o) < \infty$.

Theorem 4.3. *Let*

$$I(l_o, r] < \infty, \;\; J(l_o, r] = \infty \tag{4.15}$$

Then for $l_o < x < r$, z_i, $i = 1, 2$

$$P\left\{\tau_{x,z_i}(l_o, r] = \infty, \lim_{t\uparrow\tau_{x,z_i}(l_o,r]} X_{x,z_i}(t) = l_o\right\} = P\left\{\lim_{t\uparrow\tau_{x,z_i}(l_o,r]} X_{x,z_i}(t) = l_o\right\} = p_i(x; l_o, r) \tag{4.16}$$

i.e., provided (4.15) the left end l_o can be attained only in infinite time.

Proof. Suppose the contrary, i.e., for some $\bar{x}$, z_i, $l_o < \bar{x} < r$, $i = 1, 2$ there exist $T > 0$ and $\delta > 0$ such that

$$P\left\{\tau_{\bar{x},z_i}(l_o, r] \le T, \lim_{t\uparrow\tau_{\bar{x},z_i}(l_o,r]} X_{\bar{x},z_i}(t) = l_o\right\} \ge \delta \tag{4.17}$$

The theorem will be proved if we show that from (4.17) it follows $E\tau_{\bar{x},z_i}(l_o, r] < \infty$.

At the beginning let us prove the following assertion.

If for some $T_1 > 0$ and $\delta_1 > 0$ the inequality

$$P\{\tau_{x,z_i}(l_o, r] \le T_1\} \ge \delta_1 \tag{4.18}$$

holds for all x, z_i, $l_o < x < r$, $i = 1, 2$, then $E\tau_{x,z_i}(l_o, r] < \infty$.

Indeed from (4.18) it follows

$$P\left\{\tau_{y,z_i}(l_o,r] > T_1\right\} \le 1-\delta_1$$

for all y, z_i, $l_o < y < r$, $i = 1, 2$. Hence

$$P\left\{\tau_{x,z_i}(l_o,r] > 2T_1\right\}$$

$$= \sum_{j=1}^{2}\int_{l_o}^{r} P\left\{\tau_{y,z_j}(l_o,r] > T_1\right\}\cdot P\left\{\tau_{x,z_i}(l_o,r] > T_1,\ X_{x,z_i}(T_1)\in dy,\ Z_{x,z_i}(T_1) = z_j\right\}$$

$$\le (1-\delta_1)\cdot\sum_{j=1}^{2}\int_{l_o}^{r} P\left\{\tau_{x,z_i}(l_o,r] > T_1,\ X_{x,z_i}(T_1)\in dy,\ Z_{x,z_i}(T_1) = z_j\right\}$$

$$= (1-\delta_1)\cdot\sum_{j=1}^{2}\int_{l_o}^{r} P\left\{\tau_{x,z_i}(l_o,r] > T_1\right\} \le (1-\delta_1)^2$$

Analogously it can be ascertained that

$$P\left\{\tau_{x,z_i}(l_o,r] > mT_1\right\} \le (1-\delta_1)^m$$

From here

$$E\tau_{x,z_i}(l_o,r] < \sum_{m=0}^{\infty}(m+1)T_1\cdot(1-\delta_1)^m < \infty$$

The above-mentioned assertion is proved.

Thus it remains to prove that the inequality (4.17) which holds for some fixed $\bar{x}, z_i$ implies the inequality (4.18) and it must be fulfilled for all x, z_i, $l_o < x < r$, $i = 1, 2$. Before let us note that if the inequality (4.17) holds for $z_i = z_1$ then it holds for $z_i = z_2$ a fortiori. Let it be fulfilled for $\bar{x}, z_2$. Take the point $\bar{x}, z_1$. It is not difficult to show that there exist $\bar{t} > 0$ and $\bar{\delta} > 0$ such that

$$P\left\{\tau_{\bar{x},z_1}^{\bar{x},z_2} \le \bar{t},\ X_{\bar{x},z_1}(t)\in[\bar{x},r) \text{ for all } 0\le t\le\bar{t}\right\} \ge \bar{\delta}$$

Therefore

$$P\left\{\tau_{\bar{x},z_1}(l_o,r] \le T+\bar{t},\ \lim_{t\uparrow\tau_{\bar{x},z_1}(l_o,r]} X_{\bar{x},z_1}(t) = l_o\right\} \ge \delta\cdot\bar{\delta}$$

Thus we can regard without loss of generality that (4.17) is fulfilled for z_1 and z_2 at once. But then it is fulfilled a fortiori for all x, z_i, where $l_o < x \le \bar{x}$, $i = 1, 2$. Let us consider now $\bar{x} \le x < r$. For such x the mathematical expectation $E\tau_{x,z_i}[\bar{x},r]$ is uniformly bounded by some number T_0. From here we have

$$P\left\{\tau_{x,z_i}[\bar{x},r] \le 2T_0\right\} \ge \frac{1}{2}$$

and

$$P\left\{\tau_{x,z_i}(l_o,r] \le T+2T_0\right\} \ge P\left\{\tau_{x,z_i}[\bar{x},r] \le 2T_0\right\}\cdot P\left\{\tau_{\bar{x},z_2}(l_o,r] \le T\right\} \ge \frac{1}{2}\delta$$

So the inequality is realized for all x, z_i, $l_o < x < r$, $i = 1, 2$. Theorem 4.3 is proved.

Now on the basis of Theorems 4.2 and 4.3 we are able to establish when the Cauchy problem is possible for the following system of hyperbolic equations

$$\frac{\partial u_1}{\partial t} = a_1(x)\frac{\partial u_1}{\partial x} + (b_1(x) - q_1(x))u_1 + q_1(x)u_2 + c_1(x)$$

$$\frac{\partial u_2}{\partial t} = a_2(x)\frac{\partial u_2}{\partial x} + q_2(x)u_1 + (b_2(x) - q_2(x))u_2 + c_2(x),\ l_o < x < r_o,\ t > 0 \quad (4.19)$$

with initial conditions

$$u_i(0, x) = f_i(x),\ i = 1, 2 \quad (4.20)$$

That is possible only in four cases. List them:

$$I(l_o, r] = \infty,\ \bar{I}[l, r_o) = \infty \quad (4.21)$$

$$I(l_o, r] = \infty,\ \bar{I}[l, r_o) < \infty,\ \bar{J}[l, r_o) = \infty \quad (4.22)$$

$$I(l_o, r] < \infty,\ J(l_o, r] = \infty,\ \bar{I}[l, r_o) = \infty \quad (4.23)$$

$$I(l_o, r] < \infty,\ J(l_o, r] = \infty,\ \bar{I}[l, r_o) < \infty,\ \bar{J}[l, r_o) = \infty \quad (4.24)$$

In the other cases it is necessary to pose some boundary conditions at least on one of the ends l_o and r_o.

Of course, the solution of the Cauchy problem (4.19)-(4.20) in each of the cases (4.21)-(4.24) is given by formula (2.6), where $m = 2$, $l_o < x < r_o$, $b(x, z_i) = b_i(x)$, $c(x, z_i) = c_i(x)$, $f(x, z_i) = f_i(x)$, $i = 1, 2$.

Let us examine the asymptotic behavior of the solution under $t \to \infty$ when $b_i(x) = 0$, $c_i(x) = 0$, $i = 1, 2$. In this case the solution of the problem (4.19)-(4.20) has the simple form (see (2.6))

$$u_i(t, x) = Ef(X_{x,z_i}(t)) \quad (4.25)$$

If, for definiteness, the functions $f_i(x)$, $i = 1, 2$, are continuous on $[l_o, r_o]$ and they are equal to zero at the points l_o and r_o, then in the cases (4.22)-(4.24)

$$\lim_{t\to\infty} u_i(t, x) = 0 \quad (4.26)$$

due to Theorems 4.2 and 4.3.

In the most interesting case (4.21) the asymptotic behavior of the solution depends on whether the process (X, Z) is ergodic.

5. Ergodicity and asymptotic behavior of the solution of the Cauchy problem

The case (4.21) is the only case when the process (X, Z) is recurrent (see Corollary to Theorem 4.2). Following Kolmogorov's classification of recurrent Markov chains R.Z.Khasminskii [13] has subdivided recurrent diffusion processes into positive ones and null ones. In our case the recurrent process (X, Z) (i.e., $I(l_o, r] = \infty$, $\bar{I}[l, r_o) = \infty$) is positive if for all (x, z_i), (y, z_j)

$$E\tau_{x,z_i}^{y,z_j} < \infty \quad (5.1)$$

Otherwise the recurrent process (X, Z) is null process. Remind that due to Theorem 4.2 the time $\tau_{x,z_i}^{y,z_j} < \infty$ with probability 1.

It is not difficult to prove that the condition (5.1) is fulfilled if and only if simultaneously

$$C(l_o, r] = \lim_{l \downarrow l_o} C[l, r] < \infty$$

and

$$\bar{C}[l, r_o) = \lim_{r \uparrow r_o} \bar{C}[l, r] < \infty$$

Theorem 5.1. *If $I(l_o, r] = \infty$ then $C(l_o, r] < \infty$ if and only if*

$$K(l_o, r] = \int_{l_o}^{r} m(\xi) \exp(\int_{\xi}^{r} k(\varsigma) d\varsigma) d\xi < \infty \tag{5.2}$$

If $\bar{I}[l, r_o) = \infty$ then $\bar{C}[l, r_o) < \infty$ if and only if

$$\bar{K}[l, r_o) = \int_{l}^{r_o} m(\xi) \exp(-\int_{l}^{\xi} k(\varsigma) d\varsigma) d\xi < \infty \tag{5.3}$$

The recurrent process is positive if and only if

$$\int_{l_o}^{r_o} m(\xi) \exp(-\int_{c}^{\xi} k(\varsigma) d\varsigma) d\xi < \infty \tag{5.4}$$

where $l_o < c < r_o$ (clearly, the convergence of the integrals (5.2)-(5.4) does not depend on a choice of r, l, c).

Proof. The first two assertions are proved identically and we shall prove the second one. From the formula (3.13) we have

$$\bar{C}[l, r_o) = \lim_{r \uparrow r_o} \bar{C}[l, r] = \lim_{r \uparrow r_o} \frac{\int_l^r (\frac{1}{a_1(\xi)} + k_1(\xi) \int_l^\xi m(\eta) \exp(\int_\eta^\xi k(\varsigma) d\varsigma) d\eta) d\xi}{1 + \int_l^r k_1(\xi) \exp(\int_l^\xi k(\varsigma) d\varsigma) d\xi} \tag{5.5}$$

Because $\bar{C}[l, r] = E\tau_{l, z_1}[l, r]$ then $\bar{C}[l, r]$ grows with $r \uparrow r_o$ and therefore the limit in (5.5) (finite or infinite) always exists. Since $\bar{C}[l, r]$ grows with $r \uparrow r_o$ and the denominator in (5.5) goes to ∞ with $r \uparrow r_o$ due to condition $\bar{I}[l, r_o) = \infty$, the numerator in (5.5) also must go to ∞ and consequently we have the uncertainty $\frac{\infty}{\infty}$ in (5.5). Thanks to L'Hopital's rule

$$\lim_{r \uparrow r_o} \frac{\int_l^r k_1(\xi) \int_l^\xi m(\eta) \exp(\int_\eta^\xi k(\varsigma) d\varsigma) d\eta d\xi}{1 + \int_l^r k_1(\xi) \exp(\int_l^\xi k(\varsigma) d\varsigma) d\xi} = \lim_{r \uparrow r_o} \frac{\int_l^r m(\eta) \exp(\int_\eta^r k(\varsigma) d\varsigma) d\eta}{\exp(\int_l^r k(\varsigma) d\varsigma)}$$

$$= \lim_{r \uparrow r_o} \int_l^r m(\eta) \exp(-\int_l^\eta k(\varsigma) d\varsigma) d\eta = \bar{K}[l, r_o)$$

If $\bar{K}[l, r_o) = \infty$ then the limit (5.5) all the more is equal to ∞ and hence it remains only to prove that from $\bar{K}[l, r_o) < \infty$ follows $\bar{C}[l, r_o) < \infty$ or, that is the same, that from $\bar{C}[l, r_o) = \infty$ provided $\bar{I}[l, r_o) = \infty$ follows $\bar{K}[l, r_o) = \infty$. We shall prove the last assertion by contradiction.

So let $\bar{I}[l, r_o) = \infty$, $\bar{C}[l, r_o) = \infty$ but $\bar{K}[l, r_o) < \infty$. We need the following inequality

$$\frac{1 + q_1(r) \int_l^r m(\xi) \exp(\int_\xi^r k(\varsigma) d\varsigma) d\xi}{q_1(r) \exp(\int_l^r k(\varsigma) d\varsigma)} > \bar{C}[l, r] \tag{5.6}$$

which is the simple consequence of the inequality

$$\frac{d\bar{C}[l, r]}{dr} > 0$$

From (5.6)

$$\frac{1}{q_1(r)\exp(\int_l^r k(\varsigma)d\varsigma)} > \bar{C}[l,r] - \int_l^r m(\xi)\exp(-\int_l^\xi k(\varsigma)d\varsigma)d\xi > \bar{C}[l,r] - \bar{K}[l,r_o) \quad (5.7)$$

From here in view of $\bar{C}[l,r_o) = \infty$ we have

$$\bar{C}[l,r] - \bar{K}[l,r_o) > 1$$

for all r which are sufficiently close to r_0. From (5.7) for such r we have

$$q_1(r)\exp(\int_l^r k(\varsigma)d\varsigma) < 1$$

and therefore there exists a constant $\bar{q}$ such that for all $\xi \geq l$ the following inequality

$$q_1(\xi) \leq \bar{q}\exp(-\int_l^\xi k(\varsigma)d\varsigma)$$

is fulfilled. From here

$$\int_l^r k_1(\xi)d\xi \leq \bar{q}\int_l^r \frac{1}{a_1(\xi)}\exp(-\int_l^\xi k(\varsigma)d\varsigma)d\xi \leq \bar{q}K[l,r_o)$$

i.e., the integral $\int_l^{r_o} k_1(\xi)d\xi$ is convergent. But from convergence of this integral follows easily the convergence of the integral $\bar{I}[l,r_o)$. The obtained contradiction proves the second assertion of the theorem. The last assertion follows from the fact that the relation (5.4) is fulfilled if and only if $K(l_o,r] < \infty$ and $\bar{K}[l,r_o) < \infty$. Theorem 5.1 is proved.

Theorem 5.2. *The positive process (X,Z) has a stationary measure with the density*

$$\lambda_1(x) = \frac{L}{a_1(x)}\exp(-\int_c^x k(\varsigma)d\varsigma) = \frac{1}{a_1(x)\cdot\int_{l_o}^{r_o} m(\xi)\exp(\int_\xi^x k(\varsigma)d\varsigma)d\xi}$$

$$\lambda_2(x) = -\frac{L}{a_2(x)}\exp(-\int_c^x k(\varsigma)d\varsigma) = -\frac{1}{a_2(x)\cdot\int_{l_o}^{r_o} m(\xi)\exp(\int_\xi^x k(\varsigma)d\varsigma)d\xi} \quad (5.8)$$

where the constant L is equal to

$$L = \frac{1}{\int_{l_o}^{r_o} m(\xi)\exp(-\int_c^\xi k(\varsigma)d\varsigma)d\xi}$$

Proof. The equations (2.10) for the density in our case have a form

$$-\frac{d(\lambda_1 a_1(x))}{dx} - q_1(x)\lambda_1 + q_2(x)\lambda_2 = 0$$

$$-\frac{d(\lambda_2 a_2(x))}{dx} - q_2(x)\lambda_2 + q_1(x)\lambda_1 = 0,\ l_o < x < r_o \quad (5.9)$$

and it can be immediately checked that density $\lambda_1(x)$, $\lambda_2(x)$ from (5.8) satisfies (5.9). Theorem 5.2 is proved.

Let us prove now that there is not any bounded harmonic function with the exception of constant functions if, for example, $\bar{I}[l,r_o) = \infty$ (remember that a function is harmonic if $Ef(X_{x,z_i}(t), Z_{x,z_i}(t)) = f(x, z_i)$ for all $l_o < x < r_o$ and z_i, $i = 1, 2$). Indeed, the harmonic function $f(x, z_i)$ must satisfy the equality

$$Af = 0$$

which is equivalent to the system (3.1). From (3.1) we have

$$f_1(x) - f_2(x) = C_0 \exp(\int_l^x k(\varsigma)d\varsigma), \ f_1(x) = C_1 + C_0 \int_l^x k_1(\xi) \exp(\int_l^\xi k(\varsigma)d\varsigma)d\xi \quad (5.10)$$

Since $f_1(x) = f(x, z_1)$ must be bounded we obtain $C_0 = 0$. Hence from (5.10) $f_2(x) = f_1(x) = C_1$.

The obtained property of the process (X, Z) together with existence of the stationary measure proves that the recurrent positive process (X, Z) is ergodic. In the next theorem we give some ergodic characteristics of the process (X, Z). They can be proved by the same way as in [13].

Theorem 5.3. *If*

$$I(l_o, r] = \infty, \ \bar{I}[l, r_o) = \infty, \ K(l_o, r] < \infty, \ \bar{K}[l, r_o) < \infty \quad (5.11)$$

then the process (X, Z) is ergodic and consequently for any bounded on (l_o, r_o) function (and, for definiteness, continuous) $f(x, z_i) = f_i(x)$ with probability 1

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X_{x,z_i}(s), Z_{x,z_i}(s))ds = \int_{l_o}^{r_o} (f_1(x)\lambda_1(x) + f_2(x)\lambda_2(x)dx \quad (5.12)$$

Moreover

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Ef(X_{x,z_i}(s), Z_{x,z_i}(s))ds = \lim_{t\to\infty} \frac{1}{t} \int_0^t u_i(s, x)ds$$
$$= \int_{l_o}^{r_o} (f_1(x)\lambda_1(x) + f_2(x)\lambda_2(x)dx \quad (5.13)$$

and

$$\lim_{t\to\infty} Ef(X_{x,z_i}(t), Z_{x,z_i}(t)) = \lim_{t\to\infty} u_i(t, x) = \int_{l_o}^{r_o} (f_1(x)\lambda_1(x) + f_2(x)\lambda_2(x)dx \quad (5.14)$$

Remark about null process. Let us take as initial function $f_i(x)$ for definiteness the nonnegative continuous function ($f_i(x) \geq 0$) which is not trivial ($f_i(x) \neq 0$) and has a compact support in (l_o, r_o). We already know for such a function that $\lim_{t\to\infty} u_i(t, x) = 0$, $i = 1, 2$, for a nonrecurrent process and $\lim_{t\to\infty} u_i(t, x) > 0$ for a recurrent positive process. Further, the integral $\int_0^\infty u_i(t, x)dt$ is convergent for a nonrecurrent process and is divergent for a recurrent positive process. For a recurrent null process $\lim_{t\to\infty} u_i(t, x) = 0$, $i = 1, 2$, but the integral $\int_0^\infty u_i(t, x)dt$ is divergent. These properties express those facts that a null process for the most part of time is in the neighborhoods of the ends l_o and r_o. But on the other hand because it is recurrent it spends the infinite time in the neighborhood of any point of the interval (l_o, r_o). Besides for null process there exists the stationary unbounded measure with the help of which it is possible to investigate its asymptotic behavior. However we do not consider here these properties and the other ones (see [13]) of null processes.

6. Some applications and examples

Example 1. Consider the equation

$$\frac{dX}{dt} = Z \cdot (1 - x^2)^\alpha, \quad -\infty < \alpha < \infty \tag{6.1}$$

where the Markov chain Z takes two values $z_1 = 1$ and $z_2 = -1$, $q_1(x) = q_2(x) = (1 - x^2)^\gamma$, $-\infty < \gamma < \infty$, and $X \in (-1, 1)$. We have

$$a_1(x) = (1 - x^2)^\alpha, \; a_2(x) = -(1 - x^2)^\alpha, \; k_1(x) = (1 - x^2)^{\gamma-\alpha},$$

$$k_2(x) = (1 - x^2)^{\gamma-\alpha}, \; k(x) = 0, \; m(x) = 2(1 - x^2)^{-\alpha}$$

Direct and not complicated calculations give that under

$$\gamma - \alpha \leq -1 \tag{6.2}$$

the process (X, Z) is recurrent (the case (4.21): $I(-1, 0] = \infty$, $\bar{I}[0, 1) = \infty$), under

$$\gamma - \alpha > -1, \; \alpha \geq 1 \tag{6.3}$$

the process (X, Z) is nonrecurrent but the ends of interval $(-1, 1)$ are not attainable for finite time (the case (4.24): $I(-1, 0] < \infty$, $J(-1, 0] = \infty$, $\bar{I}[0, 1) < \infty$, $\bar{J}[0, 1) = \infty$), and under

$$\gamma - \alpha > -1, \; \alpha < 1 \tag{6.4}$$

the ends of interval $(-1, 1)$ are attainable for finite time (the case: $I(-1, 0] < \infty$, $J(-1, 0] < \infty$, $\bar{I}[0, 1) < \infty$, $\bar{J}[0, 1) < \infty$). In the cases (6.2) and (6.3) it is possible to consider the Cauchy problem for the corresponding system of hyperbolic equations and in the case (6.4) we need some boundary conditions. The case (6.2) gives a recurrent positive (ergodic) process (the case: $I(-1, 0] = \infty$, $K(-1, 0] < \infty$, $\bar{I}[0, 1) = \infty$, $\bar{K}[0, 1) < \infty$) under

$$\gamma - \alpha \leq -1, \; \alpha < 1$$

and a recurrent null process (the case: $I(-1, 0] = \infty$, $K(-1, 0] = \infty$, $\bar{I}[0, 1) = \infty$, $\bar{K}[0, 1) = \infty$) under

$$\gamma - \alpha \leq -1, \; \alpha \geq 1$$

Example 2 (see [14]). Let us consider the equation

$$\frac{dX}{dt} = -cX + Z \; (c > 0) \tag{6.5}$$

where the Markov chain Z takes two values $z_1 = b$, $z_2 = -b$ $(b > 0)$, $q_1(x) = q_2(x) = q =$const> 0 and $X \in (l_o, r_o) = (-\frac{b}{c}, \frac{b}{c})$.

We have

$$a_1(x) = b - cx > 0, \; a_2(x) = -b - cx < 0, \quad -\frac{b}{c} < x < \frac{b}{c}$$

$$k_1(x) = \frac{q}{b - cx}, \; k_2(x) = \frac{q}{-b - cx}, \; k(x) = \frac{2cqx}{b^2 - c^2x^2}, \; m(x) = \frac{2b}{b^2 - c^2x^2}$$

It is not difficult to verify that the relations (5.11) are fulfilled here. Therefore the process (X, Z) defined by (6.5) is recurrent positive process and it has stationary measure (which was found in ([14])). Due to Theorem 5.2 this measure has the following density

$$\lambda_1(x) = M \cdot \frac{(b^2 - c^2x^2)^{q/c}}{b - cx}, \ \lambda_2(x) = M \cdot \frac{(b^2 - c^2x^2)^{q/c}}{b + cx}, \ \frac{1}{M} = \frac{1}{c} \cdot (2b)^{2q/c} B(\frac{q}{c}, \frac{q}{c}) \quad (6.6)$$

where B is the beta-function:

$$B(\mu, \nu) = \int_0^1 (1 - y)^{\mu - 1} y^{\nu - 1} dy, \ \mu > 0, \ \nu > 0$$

Consider the stochastic differential equation with additive noise

$$dX = -cXdt + \sigma dW(t) \quad (6.7)$$

where $W(t)$ is a standard Wiener process.

This equation has the solution with stationary measure

$$p(x) = \frac{1}{\sqrt{2\pi} \cdot \frac{\sigma}{\sqrt{2c}}} \cdot \exp(-\frac{x^2}{2 \cdot \frac{\sigma^2}{2c}}) \quad (6.8)$$

which is known as Ornstein-Uhlenbeck process.

The equation (6.5) is obtained like the equation (6.7) by acting of additive noise on equation

$$\frac{dX}{dt} = -cX$$

for which the trivial solution is asymptotically stable. Of course, the noise in (6.5) has another nature and in particular therefore the stationary measure in the case (6.5) has the bounded support. Let us take in (6.5)

$$b = \gamma\sqrt{q}, \ \gamma > 0$$

and let $q \to \infty$. With growing q the interval $(l_o, r_o) = (-\frac{\gamma}{c}\sqrt{q}, \frac{\gamma}{c}\sqrt{q})$ is growing as well. Substituting in (6.6) $b = \gamma\sqrt{q}$ we obtain the density depending on q. Denote this density by $\lambda_i(x; q)$. It turns out that

$$\lim_{q \to \infty} \lambda_i(x; q) = \frac{1}{2\sqrt{2\pi} \cdot \frac{\gamma}{\sqrt{2c}}} \cdot \exp(-\frac{x^2}{2 \cdot \frac{\gamma^2}{2c}}), \ i = 1, 2 \quad (6.9)$$

The relation (6.9) can be easily proved if we use the formula

$$B(\mu, \nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu + \nu)}, \ \mu > 0, \ \nu > 0$$

which connects beta-functions with gamma-functions and Stirling's formula

$$\lim_{s \to \infty} \frac{\Gamma(s)}{\sqrt{2\pi} e^{-s} s^{s - 1/2}} = 1$$

Therefore the limit density for X coincides with the density of the Ornstein-Uhlenbeck process under $\gamma = \sigma$.

Let us return to (6.5) where Z again takes two fixed values $z_1 = b$, $z_2 = -b$ and $X \in (-\frac{b}{c}, \frac{b}{c})$ but q_1 and q_2 depend on x. Let the dependence have a form

$$q_1(x) = q_2(x) = q(b^2 - c^2x^2)^\alpha$$

where α can take any values. If $\alpha < 0$ then the switching frequency of the process Z near the ends $-\frac{b}{c}$ and $\frac{b}{c}$ grows to infinity and if $\alpha > 0$ it decays.

Direct calculations give us that under $\alpha > 0$ we have the case (4.24) and under $\alpha \leq 0$ we obtain the recurrent positive process, i.e., under all $-\infty < \alpha < \infty$ it is possible to consider the Cauchy problem for corresponding system of hyperbolic equations.

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